Entropy bound for a charged object from the Kerr-Newman black hole

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Abstract

We derive again the upper entropy bound for a charged object by employing thermodynamics of the Kerr-Newman black hole linearised with respect to its electric charge.

Recently, Bekenstein and Mayo [1] and Hod [2] have obtained an upper entropy bound for a charged object by requiring the validity of thermodynamics of the Reissner-Nordström black hole linearised with respect to its electric charge. They have found again the proposal of Zaslaskii [3] derived in another context. The proof takes into account the general existence of an electrostatic self-force acting on a point charge in a gravitational field [4, 5, 6] which has been exactly determined for the Schwarzschild black hole [7, 8, 9].

A question occurs immediately: what is the upper entropy bound for a charged object by requiring the validity of thermodynamics of the Kerr-Newman black hole linearised with respect to its electric charge ? It is possible to answer because the electrostatic self-force for a point charge on the symmetry axis of the Kerr black hole has been previously determined [10, 11]. The purpose of this work is to derive again the entropy bound in this situation according to the method initially due to Bekenstein [12] for a neutral object falling in a Schwarzschild black hole.

The Kerr black hole is characterized by the mass m and the angular momentum per unit mass a satisfying $m^2 > a^2$. In the coordinate system (t, r, θ, φ), the Kerr metric is

$$\begin{aligned} ds^2 &= \left(1 - \frac{2mr}{\Sigma}\right) dt^2 - \frac{\Sigma}{\triangle} dr^2 - \Sigma d\theta^2 + \frac{4amr\sin^2\theta}{\Sigma} dt d\varphi \\ &\quad - \sin^2\theta \left(r^2 + a^2 + \frac{2a^2 mr \sin^2\theta}{\Sigma}\right) d\varphi^2 \end{aligned} \tag{1}$$

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with $\triangle = r^2 - 2mr + a^2$ and $\Sigma = r^2 + a^2\cos^2\theta$. The Kerr-Newman black hole linearised with respect to its electric charge q is described by metric (1) plus an electromagnetic test field having the components

$$A_t = \frac{qr}{\Sigma} \quad A_r = 0 \quad A_\theta = 0 \quad A_\varphi = -\frac{qar\sin^2\theta}{\Sigma}. \tag{2}$$

The area of the Kerr-Newman black hole has the expression

$$\mathcal{A}(m,a,q) = 4\pi\left[\left(m + \sqrt{m^2 - a^2 - q^2}\right)^2 + a^2\right] \tag{3}$$

which reduces to

$$\mathcal{A}(m,a,q) \approx 4\pi\left[2m^2 + 2m\sqrt{m^2-a^2} - \frac{q^2 m}{\sqrt{m^2-a^2}} - q^2\right] \tag{4}$$

for a Kerr-Newman black hole linearised with respect to its electric charge q. In thermodynamics of the black hole, the entropy S_{BH} is given by

$$S_{BH}(m,j,q) = \frac{1}{4}\mathcal{A}(m,j/m,q). \tag{5}$$

in terms of the thermodynamical variables m, j and q.

On the symmetry axis of metric (1), outside the outer horizon $r_+ = m + \sqrt{m^2 - a^2}$, we consider a charged object with a mass μ, an electric charge e, an entropy S and a radius R whose the own gravitational field is negligeable and the electromagnetic field generated by the charge e is a test field. By making use of a quasi-static assumption, we restrict ourselves to the case where the charged object is at rest. We suppose that its total energy $\mathcal{E}$ with respect to a stationary observer at infinity coincides with the one of a massive point charge located at $r = r_0$ and $\theta = 0$ with $r_0 > r_+$. Obviously, the dimension R of this object is taken as the proper length along the symmetry axis of metric (1). This proper length ℓ from the outer horizon to the position r_0 for $\theta = 0$ has the expression

$$\ell = \int_{r_+}^{r_0} \frac{\sqrt{r^2+a^2}}{\sqrt{r^2 - 2mr + a^2}}dr. \tag{6}$$

The total energy $\mathcal{E}$ of a massive point charge is the sum of the energy W_{mass} of the mass μ, the electrostatic energy W_{ext} of the charge e in the exterior electromagnetic field (2) and the electrostatic self-energy W_{self} of the charge e. The electrostatic self-force f^i_{self} exerted on the point charge by its self-field has the expression [10, 11]

$$f^i_{self} = \frac{e^2 m r_0}{(r_0^2 + a^2)^2}\delta^i_1 \tag{7}$$

in Fermi coordinates at the position of the point charge. We can easily determined W_{self} so that it yields self-force (7); we find

$$W_{self} = \frac{1}{2}\frac{e^2 m}{r_0^2 + a^2}. \tag{8}$$

So, the total energy $\mathcal{E}$ is given by

$$\mathcal{E} = \frac{\mu\sqrt{r_0^2 - 2mr_0 + a^2}}{\sqrt{r_0^2 + a^2}} + \frac{eqr_0}{r_0^2 + a^2} + \frac{e^2 m}{2(r_0^2 + a^2)}. \tag{9}$$

For the charged object, its last state which is possible outside the outer horizon is defined by $\ell = R$. By assuming that ℓ is small, we deduce from (6) the asymptotic form

$$\ell \sim \frac{\sqrt{2}(r_+^2 + a^2)}{(m^2 - a^2)^{1/4}}\sqrt{r_0 - r_+} \quad \text{as} \quad r_0 \to r_+. \tag{10}$$

By substituting (10) into (9), we obtain the energy $\mathcal{E}_{last}$ of the last state

$$\mathcal{E}_{last} \sim \frac{\mu R\sqrt{m^2 - a^2}}{r_+^2 + a^2} + \frac{eqr_+}{r_+^2 + a^2} + \frac{e^2 m}{2(r_+^2 + a^2)} \quad \text{as} \quad R \to 0. \tag{11}$$

We are now in a position to apply thermodynamics of the Kerr-Newman black hole when the charged object falls infinitely slowly along the symmetry axis until the absorption inside the outer horizon which is the final state. This is the original method of Bekenstein [12] for a neutral object in the Schwarzschild black hole. The final state is again a Kerr-Newman black hole but with the new parameters

$$m_f = m + \mathcal{E}_{last} \quad , \quad j_f = j \quad \text{and} \quad q_f = q + e. \tag{12}$$

In the last state outside the outer horizon, the total entropy is $S_{BH}(m, j, q) + S$ but after the absorption the entropy is only $S_{BH}(m_f, j_f, q_f)$ in the final state. By virtue of the generalised second law of thermodynamics, we must have

$$S_{BH}(m_f, j_f, q_f) \geq S_{BH}(m, j, q) + S. \tag{13}$$

We can calculate $\triangle S_{BH} = S_{BH}(m_f, j_f, q_f) - S_{BH}(m, j, q)$ from expression (4) with increments (12) by keeping only linear terms in $\mathcal{E}_{last}$. We find

$$\triangle S_{BH} = \frac{2\pi}{\sqrt{m^4 - j^2}}\left[2m\left(m^2 + \sqrt{m^4 - j^2}\right)\mathcal{E}_{last} - \left(m^2 + \sqrt{m^4 - j^2}\right)\left(eq + \frac{1}{2}e^2\right)\right] \tag{14}$$

that we can rewrite under the form

$$\triangle S_{BH} = \frac{2\pi}{\sqrt{m^2 - a^2}}\left[(r_+^2 + a^2)\mathcal{E}_{last} - eqr_+ - \frac{1}{2}e^2 r_+\right]. \tag{15}$$

We simplify expression (15) by using $\mathcal{E}_{last}$ given by (11). Then, taking into account inequality (13), we thus obtain the desired entropy bound

$$S \leq 2\pi\left(\mu R - \frac{1}{2}e^2\right). \tag{16}$$

In conclusion, we have extended the works of Bekenstein and Mayo [1] and Hod [2] by employing thermodynamics of the Kerr-Newman black hole instead of the Reissner-Nordström black hole. These kinds of method for determining the entropy bound is without pretending to any rigour. However, they confirm the physical importance of the electrostatic self-force acting on a point charge in a background black hole although its physical relation with the entropy of a charged object is not clear.

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